On global location-domination in bipartite graphs

Carmen Hernando, Merce Mora, Ignacio M. Pelayo

Universitat Politècnica de Catalunya, Barcelona, Spain

Abstract

A dominating set S of a graph G is called *locating-dominating*, *LD-set* for short, if every vertex v not in S is uniquely determined by the set of neighbors of v belonging to S. Locating-dominating sets of minimum cardinality are called *LD*-codes and the cardinality of an LD-code is the *location-domination number* $\lambda(G)$. An LD-set S of a graph G is *global* if it is an LD-set of both G and its complement $\overline{G}$. The *global location-domination number* $\lambda_g(G)$ is the minimum cardinality of a global LD-set of G.

For any LD-set S of a given graph G, the so-called *S-associated graph* G^S is introduced. This edge-labeled bipartite graph turns out to be very helpful to approach the study of LD-sets in graphs, particularly when G is bipartite.

This paper is mainly devoted to the study of relationships between global LD-sets, LD-codes and the location-domination number in a graph G and its complement $\overline{G}$, when G is bipartite.

Keywords: Domination, Global domination, Locating domination, Complement graph, Bipartite graph.

1 Introduction

Let $G = (V, E)$ be a simple, finite graph. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$. The *complement* of a graph G, denoted by $\overline{G}$, is the graph on the same vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The distance between vertices $v, w \in V$ is denoted by $d_G(v, w)$. We write $N(u)$ or $d(v, w)$ if the graph G is clear from the context. Given any pair of sets A and B, $A \triangle B$ denotes its symmetric difference, that is, $(A \setminus B) \cup (B \setminus A)$. For further notation and terminology , we refer the reader to [6].

A set $D \subseteq V$ is a *dominating set* if for every vertex $v \in V \setminus D$, $N(v) \cap D \neq \emptyset$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G [8]. A dominating set is *global* if it is a dominating set of both G and its complement graph, $\overline{G}$. The minimum cardinality of a global dominating set of G, denoted by $\gamma_g(G)$, is the *global domination number* of G [3, 4, 14]. If D is a subset of V and $v \in V \setminus D$, we say that v *dominates* D if $D \subseteq N(v)$.

A dominating set $S \subseteq V$ is a *locating-dominating set*, *LD-set* for short, if for every two different vertices $u, v \in V \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The *location-domination number* of G, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is called an *LD-code* [13, 15]. Certainly, every LD-set of a non-connected graph G is the union of LD-sets of its connected components and the location-domination number is the sum of the location-domination number of its connected components. LD-codes and the location-domination parameter have been intensively studied during the last decade; see [1, 2, 5, 7, 11, 9]. A complete and regularly updated list of papers on locating-dominating codes is to be found in [12].

The remaining part of this paper is organized as follows. In Section 2, we deal with the problem of approaching the relationship between $\lambda(G)$ and $\lambda(\overline{G})$, for any arbitrary graph G. In Section 3, we introduce the so-called *LD-set-associated graph* G^S, which is an edge-labeled bipartite graph constructed from an arbitrary LD-set S of a given graph G, and show some basic properties of this graph. Finally, Section 4 is concerned with the study of relationships between the location-domination number $\lambda(G)$ of a bipartite graph G and the location-domination number $\lambda(\overline{G})$ of its complement $\overline{G}$.

2 General case

This section is devoted to approach the relationship between $\lambda(G)$ and $\lambda(\overline{G})$, for any arbitrary graph G. Some of the results we present were previously shown in [9, 10] and we include them for the sake of completeness.

Notice that $N_{\overline{G}}(x) \cap S = S \setminus N_G(x)$ for any set $S \subseteq V$ and any vertex $x \in V \setminus S$. A straightforward consequence of this fact are the following results.

Proposition 1 ([10]). *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then S is an LD-set of $\overline{G}$ if and only if S is a dominating set of $\overline{G}$.*

Proposition 2 ([9]). *Let $S \subseteq V$ be an LD-set of a graph $G = (V, E)$. Then, the following holds.*

(a) There is at most one vertex $u \in V \setminus S$ dominating S, and in the case it exists, $S \cup \{u\}$ is an LD-set of $\overline{G}$.

(b) S is an LD-set of $\overline{G}$ if and only if there is no vertex in $V \setminus S$ dominating S in G.

The following theorem is a consequence of the preceding propositions.

Theorem 1 ([9]). *For every graph G, $|\lambda(G) - \lambda(\overline{G})| \leq 1$.*

According to the preceding inequality, for every graph G, $\lambda(\overline{G}) \in \{\lambda(G) - 1, \lambda(G), \lambda(G) + 1\}$, all cases being feasible for some connected graph G. See Table 1 for some basic examples covering all possible cases.

We intend to obtain either necessary or sufficient conditions for a graph G to satisfy $\lambda(\overline{G}) > \lambda(G)$, i.e., $\lambda(\overline{G}) = \lambda(G) + 1$. This problem was approached and completely solved in [10] for the family of block-cactus. In this work, we carry out a similar study for bipartite graphs. After noticing that solving the equality $\lambda(\overline{G}) = \lambda(G) + 1$ is closely related to analyzing the existence or not of sets that are simultaneously locating-dominating sets in both G and its complement $\overline{G}$, the following definitions were introduced in [10].

Definition 1 ([10]). A set S of vertices of a graph G is a *global LD-set* if S is an LD-set of both G and its complement $\overline{G}$. The *global location-domination number* of a graph G, denoted by $\lambda_g(G)$, is defined as the minimum cardinality of a global LD-set of G.

According to Proposition 2, an LD-set S of a graph G is non-global if and only if there exists a (unique) vertex $u \in V(G) \setminus S$ which dominates S, i.e., such that $S \subseteq N(u)$. Notice that, for every graph G, $\lambda_g(\overline{G}) = \lambda_g(G)$, since for every set of vertices $S \subset V(G) = V(\overline{G})$, S is a global LD-set of G if and only if it is a global LD-set of $\overline{G}$. Observe also that an LD-code S of G is a global LD-set if and only if it is both an LD-code of G and an LD-set of $\overline{G}$.

Theorem 2 ([10]). *For any graph $G = (V, E)$, $\max\{\lambda(G), \lambda(\overline{G})\} \leq \lambda_g(G) \leq \min\{\lambda(G) + 1, \lambda(\overline{G}) + 1\}$. Moreover,*

(a) If $\lambda(G) \neq \lambda(\overline{G})$, then $\lambda_g(G) = \max\{\lambda(G), \lambda(\overline{G})\}$.

(b) If $\lambda(G) = \lambda(\overline{G})$, then $\lambda_g(G) \in \{\lambda(G), \lambda(G) + 1\}$, and both possibilities are feasible.

(c) $\lambda_g(G) = \lambda(G) + 1$ if and only if every LD-code of G is non-global.

Corollary 1. *If G is a graph with a global LD-code, then $\lambda(\overline{G}) \leq \lambda(G)$.*

In Table 1, the location-domination number of some families of graphs is displayed, along with the location-domination number of its complement graphs and the global location-domination number. Concretely, we consider the path P_n of order $n \geq 7$; the cycle C_n of order $n \geq 7$; the wheel W_n of order $n \geq 8$, obtained by joining a new vertex to all vertices of a cycle of order $n-1$; the complete graph K_n of order $n \geq 2$; the complete bipartite graph $K_{r,n-r}$ of order $n \geq 4$, with $2 \leq r \leq n-r$ and stable sets of order r and $n-r$, respectively; the star $K_{1,n-1}$ of order $n \geq 4$, obtained by joining a new vertex to $n-1$ isolated vertices; and finally, the bi-star $K_2(r, s)$ of order $n \geq 6$ with $3 \leq r \leq s = n - r$, obtained by joining the central vertices of two stars $K_{1,r-1}$ and $K_{1,s-1}$ respectively.

Proposition 3 ([10]). *Let G be a graph of order n. If $G \in \{P_n, C_n, W_n, K_n, K_{1,n-1}, K_{r,n-r}, K_2(r, s)\}$, then the values of $\lambda(G)$, $\lambda(\overline{G})$ and $\lambda_g(G)$ are known and they are displayed in Table 1.*

G	P_n	C_n	W_n	K_n	$K_{1,n-1}$	$K_{r,n-r}$	$K_2(r,s)$
n	$n \geq 7$	$n \geq 7$	$n \geq 8$	$n \geq 2$	$n \geq 4$	$2 \leq r \leq n-r$	$3 \leq r \leq s$
$\lambda(G)$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$n-1$	$n-1$	$n-2$	$n-2$
$\lambda(\overline{G})$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n+1}{5} \rceil$	n	$n-1$	$n-2$	$n-3$
$\lambda_g(G)$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n+1}{5} \rceil$	n	$n-1$	$n-2$	$n-2$

Table 1: The values of $\lambda(G)$, $\lambda(\overline{G})$ and $\lambda_g(G)$ for some families of graphs.

3 The LD-set-associated graph

Let S be an LD-set of a graph G. We introduce in this section a labeled graph associated to S and study some general properties. This graph will allow us to derive some properties related to LD-sets and the location-domination number of G.

Definition 2. Let S be an LD-set with exactly k vertices of a connected graph $G = (V, E)$ of order n. Consider $z \notin V(G)$ and define $N_G(z) = \emptyset$. The so-called *S-associated graph*, denoted by G^S, is the edge-labeled graph defined as follows.

(1) $V(G^S) = (V \setminus S) \cup \{z\}$;

(2) For every pair of vertices $x, y \in V(G^S)$, $xy \in E(G^S)$ if and only if $|(N_G(x) \cap S) \triangle (N_G(y) \cap S)| = 1$;

(3) The label $\ell(xy)$ of edge $xy \in E(G^S)$ is the only element of $(N_G(x) \cap S) \triangle (N_G(y) \cap S) \in S$.

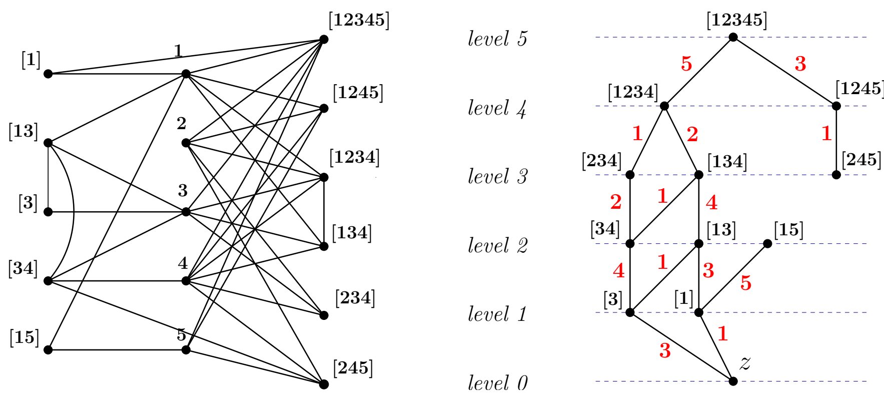


Figure 1: Left: a graph G. Right: the LD-set-associated graph G^S, where $S = \{1, 2, 3, 4, 5\}$.

Notice that two vertices of $V \setminus S$ are adjacent in G^S if their neighborhood in S differ in exactly one vertex, the label of the edge, and z is adjacent to vertices of $V \setminus S$ with exactly a neighbor in S. Therefore, we can represent the graph G^S with the vertices lying on $|S| + 1$ levels, from bottom (level 0) to top (level $|S|$), in such a way that vertices with exactly k neighbors in S are at level k. There is at most one vertex at level $|S|$ and, if it is so, this vertex is adjacent to all vertices of S. The vertices at level 1 are those with exactly one neighbor in S and z is the unique vertex at level 0. An edge of G^S has its endpoints at consecutive levels. Moreover, if $e = xy \in E(G^S)$, with $\ell(e) = u \in S$, and x is at exactly one level higher than y, then $N(x) \cap S = (N(y) \cap S) \cup \{u\}$, i.e., x and y have the same neighborhood in $S \setminus \{u\}$. Therefore, the existence of an edge in G^S with label $u \in S$ means that $S \setminus \{u\}$ is not an LD-set. Hence, if S is an LD-code, then for every $u \in S$ there exists at least an edge in G^S with label u. See Figure 1 for an example of an LD-set-associated graph.

The following proposition states some properties of LD-set-associated graphs.

Proposition 4. *Let S be an LD-set with exactly k vertices of a connected graph $G = (V, E)$ of order n. Let G^S be its S-associated graph. Then the following holds.*

1. $|V(G^S)| = n - k + 1$.

2. *G^S is bipartite.*

3. *Incident edges have different labels.*

4. *Every cycle of G^S contains an even number of edges labeled v, for all $v \in S$.*

5. *Let ρ be a walk with no repeated edges in G^S. If ρ contains an even number of edges labeled v for every $v \in S$, then ρ is a closed walk.*

6. *If $\rho = x_i x_{i+1} \dots x_{i+h}$ is a path satisfying that vertex x_{i+h} lies at level $i+h$, for any $h \in \{0, 1, \dots, h\}$, then*

   (a) the edges of ρ have different labels;

   (b) for all $j \in \{i+1, i+2, \dots, i+h\}$, $N(x_j) \cap S$ contains the vertex $\ell(x_k x_{k+1})$, for any $k \in \{i, i+1, \dots, j-1\}$.

Proof. 1. It is a direct consequence from the definition of G^S.

2. Consider the sets $V_1 = \{x \in V(G^S) : |N(x) \cap S| \text{ is odd}\}$ and $V_2 = \{x \in V(G^S) : |N(x) \cap S| \text{ is even}\}$. Then $V(G^S) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Since $||N(x) \cap S| - |N(y) \cap S|| = 1$ for any $xy \in E(G^S)$, it is clear that the vertices x, y are not in the same subset V_i, $i = 1, 2$.

3. Suppose that edges $e_1 = xy$ and $e_2 = yz$ have the same label $l(e_1) = l(e_2) = v$. This means that the sets $N(x) \cap S$ and $N(y) \cap S$ differ only in element v and the sets $N(y) \cap S$ and $N(z) \cap S$ differ only in element $v \in S$. It is only possible if $N(x) \cap S = N(z) \cap S$, implying that $x = z$.

4. Let ρ be a cycle such that $E(\rho) = \{x_0 x_1, x_1 x_2, \dots x_h x_0\}$. The set of neighbors in S of two consecutive vertices differ exactly in one vertex. If we begin with $N(x_0) \cap S$, each time we add (remove) the vertex of the label of the corresponding edge, we have to remove (add) it later in order to obtain finally the same neighborhood, $N(x_0) \cap S$. Therefore, ρ contains an even number of edges with label v.

5. Consider the vertices $x_0, x_1, x_2, x_3, ..., x_{2k}$ of the walk ρ. In this case, $N(x_{2k}) \cap S$ is obtained from $N(x_0) \cap S$ by adding or removing the labels of all the edges of the walk. Since every label appears an even number of times, for each element $v \in S$ we can match its appearances in pairs, and each pair means that we add and remove (or remove and add) it from the neighborhood in S. Therefore, $N(x_{2k}) \cap S = N(x_0) \cap S$, and hence $x_0 = x_{2k}$.

6. It straightly follows from the fact that $N(x_j) \cap S = N(x_{j-1} \cap S) \cup \{\ell(x_{j-1} x_j)\}$, for any $j \in \{i+1, \dots, i+h\}$. □

4 The bipartite case

In the sequel, $G = (V, E)$ stands for a bipartite connected graph of order $n = r + s \geq 4$, such that $V = U \cup W$, being U,W their stable sets and $1 \leq |U| = r \leq s = |W|$.

This section is devoted to solving the equation $\lambda(\overline{G}) = \lambda(G) + 1$ when we restrict ourselves to bipartite graphs. According to Corollary 1, this equality is feasible only for graphs without global LD-codes.

Lemma 1. *Let S be an LD-code of G. Then, $\lambda(\overline{G}) \leq \lambda(G)$ if any of the following conditions holds.*

1. *$S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$.*

2. $r < s$ *and* $S = W$.

3. $2^r \leq s$.

Proof. If S satisfies item 1., then there is no vertex dominating S and, by Proposition 2, S is a global LD-code of G, which, according to Corollary 1, means that $\lambda(\overline{G}) \leq \lambda(G)$. Next, assume that $r < s$ and $S = W$. In this case, U is not an LD-set, but is a dominating set since G is connected. Therefore, there exists a pair of vertices $w_1, w_2 \in W$ such that $N(w_1) = N(w_2)$. Hence, $W - \{w_1\}$ is an LD-set of $G - w_1$. Let $u \in U$ be a vertex adjacent to w_1 (it exists since G is connected), and notice that $(W \setminus \{w_1\}) \cup \{u\}$ is an LD-code of G with vertices in both stable sets, which, by the preceding item, means that $\lambda(\overline{G}) \leq \lambda(G)$. Finally, if $2^r \leq s$ then $S \neq U$, which means that S satisfies either item 1. or item 2. $\square$

Corollary 2. *If* $\lambda(\overline{G}) = \lambda(G) + 1$, *then* $r \leq s \leq 2^r - 1$. *Moreover, if* $r < s$ *then* U *is the unique LD-code of* G, *and if* $r = s$ *we may assume that* U *is a non-global LD-code of* G.

Proposition 5. *If* G *has order at least 3 and* $1 \leq r \leq 2$, *then* $\lambda(\overline{G}) \leq \lambda(G)$.

Proof. If $r = 1$, then G is the star $K_{1,n-1}$ and $\lambda(\overline{G}) = \lambda(G) = n - 1$.

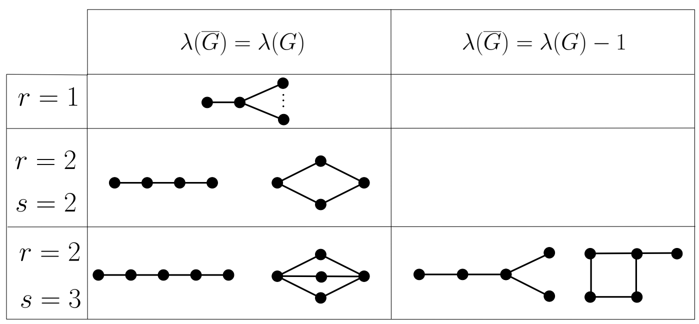


Figure 2: Some bipartite graphs with $1 \leq r \leq 2$.

Suppose that $r = 2$. If $s \geq 2^2 = 4$ then, by Lemma 1, $\lambda(\overline{G}) \leq \lambda(G)$.

If $s = 2$, then G is either P_4 and $\lambda(\overline{P_4}) = \lambda(P_4) = 2$, or G is C_4 and $\lambda(\overline{C_4}) = \lambda(C_4) = 2$.

If $s = 3$, then G is P_5, $K_{2,3}$, $K_2(1,2)$, or a banner P, and $\lambda(\overline{P_5}) = \lambda(P_5) = 2$, $\lambda(\overline{K_{2,3}}) = \lambda(K_{2,3}) = 3$, $2 = \lambda(\overline{K_2(1,2)}) < \lambda(K_2(1,3)) = 3$, and $2 = \lambda(\overline{P}) < \lambda(P) = 3$. $\square$

Notice that the only bipartite graphs G such that $\lambda(G) = 2$ are P_3, P_4, C_4 and P_5. Observe also that every bipartite graph G such that $\lambda(\overline{G}) = \lambda(G) + 1$ satisfies $\lambda(G) \geq r$, being r the order of its smallest stable set.

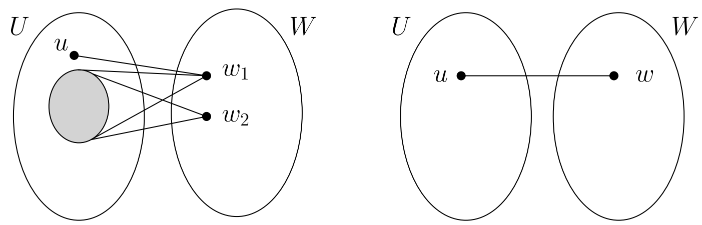
Next, we approach the case $\lambda(G) \geq 3$. That is to say, from now on we assume that $r \geq 3$.

Lemma 2. *If* $\lambda(\overline{G}) = \lambda(G) + 1$ *and* U *is an LD-code of* G, *then* G^U *contains, for every vertex* $u \in U$, *at least two edges with label* u.

Proof. Condition $\lambda(\overline{G}) = \lambda(G)+1$ implies that there is no LD-code of G with vertices in both stable sets. Therefore, for any $u \in U$, $U \setminus \{u\}$ is not an LD-set of the graph $G-u$, otherwise the set $U \setminus \{u\}$ together with a neighbor of vertex u would be an LD-code of G with vertices in both stable sets. We distinguish two possible cases.

Case (a). If $N(U \setminus \{u\}) = W$ there is at least a pair of vertices $w_1, w_2 \in W$ such that $N(w_1) \triangle N(w_2) = \{u\}$ (see Figure 3,(a)). Moreover, since there is no LD-code with vertices in both stable sets, there must be another pair of vertices $w_3, w_4 \in W$ such that $N(w_3) \triangle N(w_4) = \{u\}$, otherwise $(U \setminus \{u\}) \cup \{w\}$, where w is the neighbor of u in $\{w_1, w_2\}$, would be an LD-code with vertices in both stable sets.

Case (b). If $N(U \setminus \{u\}) \subsetneq W$, then there is exactly a vertex $w \in W$ such that $N(w) = \{u\}$ (see Figure 3,(b)). By the other hand, if the neighborhood in $U \setminus \{u\}$ of any two vertices of $W \setminus \{w\}$ is different, then $(U \setminus \{u\}) \cup \{w\}$ would be an LD-code with vertices in both stable sets. Therefore, there is at least a pair of vertices $w_1, w_2 \in W \setminus \{w\}$ such that $N(w_1) \triangle N(w_2) = \{u\}$. Notice that in this case $N(w) \triangle \emptyset = \{u\}$.



(a) $N(w_1) \triangle N(w_2) = \{u\}$ (b) $N(w) \triangle \emptyset = \{u\}$

Figure 3: Case (a): $N(U \setminus \{u\}) = W$. Case (b): $N(U \setminus \{u\}) \subsetneq W$.

Consequently, in both cases, for every $u \in U$, there are at least two edges with label u in the graph G^U. $\square$

In the study of LD-sets using the LD-associated graph, a family of graphs is particularly useful, the *cactus graph* family. A *block* of a graph is a maximal connected subgraph with no cut vertices. A connected graph G is a *cactus* if all its blocks are cycles or edges. Cactus are characterized as those connected graphs with no edge shared by two cycles.

Lemma 3. *Let $\lambda(\overline{G}) = \lambda(G) + 1$ and assume that U is an LD-code of G. Consider a subgraph H of G^U induced by a set of edges containing exactly two edges with label u, for each $u \in U$. Then, all connected components of H are cactus.*

Proof. We will prove that there is no edge lying on two different cycles of H. Suppose on the contrary that there is an edge e_1 contained in two different cycles C_1 and C_2 of H. If the label of e_1 is $u \in U$, by Proposition 4 both cycles C_1 and C_2 contain the other edge e_2 of H labeled with u. Suppose that $e_1 = x_1y_1$ and $e_2 = x_2y_2$ and assume w.l.o.g. that there exist $x_1 - x_2$ and $y_1 - y_2$ paths in C_1 not containing edges e_1, e_2. Let P_1 and P_1' denote respectively those paths (see Figure 4 a).

We have two possibilities for C_2: (i) there are $x_1 - x_2$ and $y_1 - y_2$ paths in C_2 not containing neither e_1 nor e_2. Let P_2 denote the $x_1 - x_2$ path in C_2 in that case (see Figure 4 b); (ii) there are $x_1 - y_2$ and $y_1 - x_2$ paths in C_2 not containing neither e_1 nor e_2 (see Figure 4 c).

In case (ii), the closed walk formed with the path P_1, e_1 and the $y_1 - x_2$ path in C_2 would contain a cycle with exactly an edge labeled with u, which is a contradiction (see Figure 4 d).

In case (i), at least one the following cases holds: the $x_1 - x_2$ paths in C_1 and in C_2, P_1 and P_2, are different or the $y_1 - y_2$ paths in C_1 and in C_2 are different (otherwise, $C_1 = C_2$).

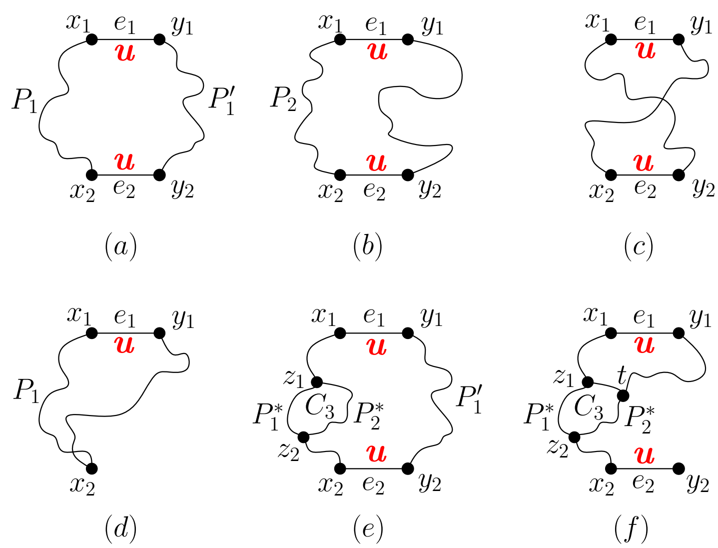


Figure 4: All connected components of the subgraph H are cactus.

Assume that P_1 and P_2 are different. Let z_1 be the last vertex shared by P_1 and P_2 advancing from x_1 and let z_2 be the first vertex shared by P_1 and P_2 advancing from z_1 in P_2. Notice that $z_1 \neq z_2$. Consider the cycle C_3 formed with the $z_1 - z_2$ paths in P_1 and P_2. Let P_1^* and P_2^* be respectively the $z_1 - z_2$ subpaths of P_1 and P_2 (see Figure 4 e). We claim that the internal vertices of P_2^* do not lie in P_1'. Otherwise, consider the first vertex t of P_1' lying also in P_2^*. The cycle beginning in x_1, formed by the edge e_1, the $y_1 - t$ path contained in P_1', the $t - z_1$ path contained in P_2^* and the $z_1 - x_1$ path contained in P_1 has exactly one appearance of an edge with label u, which is a contradiction (see Figure 4 f). By Proposition 4, the labels of edges belonging to P_1^* appear exactly two times in cycle C_3, but they also appear exactly two times in cycle C_1. But this is only possible if they appear exactly two times in P_1^*, since H contains exactly to edges with the same label. By Proposition 4, P_1^* must be a closed path, which is a contradiction. $\square$

We present next some properties relating parameters of bipartite graphs having cactus as connected components.

Lemma 4. *Let H be a bipartite graph of order at least 4 such that all its connected components are cactus. If H has $cc(H)$ connected components and $cy(H)$ cycles, then the following holds.*

1. $|V(H)| = |E(H)| - cy(H) + cc(H)$.
2. *If* $ex(H) = |E(H)| - 4\,cy(H)$, *then* $ex(H) \geq 0$ *and* $|V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}ex(H) + cc(H)$.
3. $|V(H)| \geq \frac{3}{4}|E(H)| + 1$.
4. $|V(H)| = \frac{3}{4}|E(H)| + 1$ *if and only if* H *is connected and all blocks are cycles of order 4.*

Proof.

1. Since H is a planar graph with $\mathrm{cy}(H)+1$ faces and $\mathrm{cc}(H)$ connected components, the equality follows from the generalization of Euler's Formula:

$$(\mathrm{cy}(H) + 1) + |V(H)| = |E(H)| + (\mathrm{cc}(H) + 1).$$

2. All cycles of a bipartite graph have at least 4 edges, hence $\mathrm{ex}(H) \geq 0$. By the preceding item,

$$|V(H)| = |E(H)| - \mathrm{cy}(H) + \mathrm{cc}(H) = |E(H)| - \frac{1}{4}(|E(H) - \mathrm{ex}(H)) + \mathrm{cc}(H) = \frac{3}{4}|E(H)| + \frac{1}{4}\mathrm{ex}(H) + \mathrm{cc}(H).$$

3. It immediately follows from the preceding item.

4. Observe first that if H is connected and all blocks are cycles of order 4, then $\mathrm{cc}(H) = 1$ and $|E(H)| = 4\,\mathrm{cy}(H)$. Hence, $\mathrm{ex}(H) = |E(H)| - 4\,\mathrm{cy}(H) = 0$ and by item 2, $|V(H)| = \frac{3}{4}|E(H)| + 1$. Conversely, suppose that $|V(H)| = \frac{3}{4}|E(H)| + 1$. The graph H must be connected, since otherwise $|V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\mathrm{ex}(H) + \mathrm{cc}(H) \geq \frac{3}{4}|E(H)| + 2$. On the other hand, if H contains a cycle of order at least 6 or a bridge, then $\mathrm{ex}(H) = |E(H)| - 4\,\mathrm{cy}(H) > 0$, implying that $|V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\mathrm{ex}(H) + \mathrm{cc}(H) > \frac{3}{4}|E(H)| + \mathrm{cc}(H) = \frac{3}{4}|E(H)| + 1$. □

Proposition 6. *If* $r \geq 3$ *and* $\lambda(\overline{G}) = \lambda(G) + 1$, *then* $\frac{3r}{2} \leq s \leq 2^r - 1$.

Proof. By Corollary 2, we have that $s \leq 2^r - 1$, and we may assume that U is a non-global LD-code and there is no LD-code with vertices in both stable sets.

Consider a subgraph H of G^U with exactly two edges with label u for any $u \in U$. The graph H is bipartite since it is a subgraph of G^U and by Lemma 4,

$$s + 1 = |V(G^U)| \geq |V(H)| \geq \frac{3}{4}\,|E(H)| + 1 = \frac{3}{4}\,(2r) + 1 = \frac{3r}{2} + 1$$

and consequently $s \geq \frac{3r}{2}$. □

Lemma 5. *If* $\lambda(\overline{G}) = \lambda(G) + 1$ *and* U *is an LD-code of* G, *let* z *be the vertex of* G^U *introduced in Definition 2 and let* H *be a subgraph of* G^U *with exactly two edges with label* u, *for each* $u \in U$. *Then the following holds.*

1. *If* H *has at least two connected components, then* $s \geq \frac{3r}{2} + 1$.
2. *If* $\deg_{G^U}(z) = 0$, *then* $s \geq \frac{3r}{2} + 1$.
3. $\deg_{G^U}(z) \neq 0$ *if and only if there is at least a vertex in* W *of degree 1 in* G.
4. *If* G *has no vertex of degree 1 in* W, *then* $s \geq \frac{3r}{2} + 1$.

Proof. 1. By Lemma 4, $s+1 \geq |V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\mathrm{ex}(H) + \mathrm{cc}(H) \geq \frac{3}{4}|E(H)| + 2 = \frac{3r}{2} + 2$, and thus, $s \geq \frac{3r}{2} + 1$.

2. If $\deg_{G^U}(z) = 0$, then z is not a vertex of H. Hence, $s \geq |V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\mathrm{ex}(H) + \mathrm{cc}(H) \geq \frac{3}{4}|E(H)| + 1 = \frac{3r}{2} + 1$.

3. We know that $\deg_{G^U}(z) \neq 0$ if and only if there is a vertex $w \in W$ satisfying $N(w) \triangle N(z) = N(w) \triangle \emptyset = \{u\}$, i.e. if and only if $\deg_G(w) = 1$.

4. It is a straight consequence of items 2 and 3. $\square$

Proposition 7. *There are no bipartite graphs G satisfying $\lambda(\overline{G}) = \lambda(G) + 1$ if $\frac{3r}{2} \leq s < \frac{3r}{2} + 1$.*

Proof. Suppose on the contrary that G is a bipartite graph satisfying the conditions of the proposition. Condition $\lambda(\overline{G}) = \lambda(G) + 1$ implies that we may assume that U is an LD-code of G, there is no LD-code with vertices in both stable sets and U is not an LD-set of $\overline{G}$. Consider a subgraph H of G^U with exactly two edges with label u, for each $u \in U$ (it exists by Lemma 2).

Observe that the inequality is only possible for $s = \frac{3r}{2}$, whenever r is even, and for $s = \frac{3r+1}{2}$, whenever r is odd. If r is even and $s = \frac{3r}{2}$, then

$$\frac{3r}{2} + 1 = s + 1 = |V(G^U)| \geq |V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\mathrm{ex}(H) + \mathrm{cc}(H) = \frac{3r}{2} + \frac{1}{4}\mathrm{ex}(H) + \mathrm{cc}(H).$$

Since $\mathrm{ex}(H) \geq 0$ and $\mathrm{cc}(H) \geq 1$, this is only possible for $\mathrm{ex}(H) = 0$, $\mathrm{cc}(H) = 1$, and $V(G^U) = V(H)$. By Lemma 4, H is a cactus with all blocks cycles of order 4, concretely, $\frac{r}{2}$ cycles. If r is odd and $s = \frac{3r+1}{2}$, then

$$\frac{3r}{2} + \frac{2}{4} + 1 = \frac{3r+1}{2} + 1 = s + 1 = |V(G^U)| \geq |V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\mathrm{ex}(H) + \mathrm{cc}(H) = \frac{3r}{2} + \frac{1}{4}\mathrm{ex}(H) + \mathrm{cc}(H).$$

This is only possible for $\mathrm{ex}(H) = 2$, $\mathrm{cc}(H) = 1$, and $V(G^U) = V(H)$. By Lemma 4, H is a cactus with exactly $\frac{r-1}{2}$ cycles: $\frac{r-1}{2} - 1$ cycles of order 4 and a cycle of order 6, or $\frac{r-1}{2}$ cycles of order 4 and two bridges.

We also know that condition $\lambda(\overline{G}) = \lambda(G)$ implies the existence of a vertex $w^* \in V(G) \subseteq V(G^U) = V(H)$ such that $N_G(w^*) = U$, i.e., H has a vertex at the highest level. Lemma 5 allows us to conclude that H is connected and $z \in V(H)$. Thus, H must be a chain of cycles of order 4, or a chain of a cycle of order 6 and cycles of order 4, or a chain of a bridge and cycles of order 4, plus another bridge hanging from a vertex of this chain, with both bridges having the same label and, by Proposition 4, not lying in a path with all vertices at different levels (see Figure 5).

In consequence, one of the following cases holds in H: (i) z belongs to a cycle C of order 4; (ii) z belongs to a cycle C of order 6; (iii) z belongs to a bridge, e. In this case, there is no $x-z$ path of length i in H with consecutive vertices in levels $i, i-1, \ldots, 1, 0$ respectively containing both edges of H with label $\ell(e)$. We may assume w.l.o.g. that the labels $a, b, c \in U$ of the edges of C and e are those of Figure 6. Let w_0 be the vertex of G indicated in the same figure.

We claim that the set $S = (U \setminus \{a\}) \cup \{w_0\}$ is an LD-set of $\overline{G}$ with exactly r vertices. Indeed, if $w_0 \neq w^*$, then $N_{\overline{G}}(a) \cap S = S \setminus \{w_0\}$, $N_{\overline{G}}(w^*) \cap S = \{w_0\}$ and for any $x \in W \setminus \{w^*, w_0\}$, $N_{\overline{G}}(x) \cap S = \{w_0\} \cup S'$, where $S' = U \setminus (N_G(x) \cup \{a\}) \neq \emptyset$, since $N_G(x) \neq U \setminus \{a\}$. Moreover, for

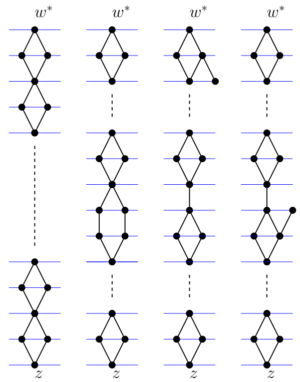


Figure 5: Examples of subgraphs H.

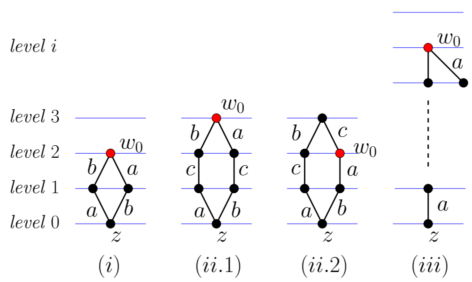


Figure 6: Possible cases for vertex z in subgraph H.

any pair of different vertices $x, y \in W \setminus \{w^*, w_0\}$, $N_G(x) \cap (U \setminus \{a\}) \neq N_G(y) \cap (U \setminus \{a\})$, implies that $N_{\overline{G}}(x) \cap S \neq N_{\overline{G}}(y) \cap S$. If $w_0 = w^*$, then $N_{\overline{G}}(a) \cap S = S \setminus \{w^*\}$, and for any $x \in W \setminus \{w^*\}$, $N_{\overline{G}}(x) \cap S = \{w^*\} \cup S'$, where $S' = U \setminus (N_G(x) \cup \{a\})$. Moreover, for any pair of different vertices $x, y \in W \setminus \{w^*\}$, $N_G(x) \cap (U \setminus \{a\}) \neq N_G(y) \cap (U \setminus \{a\})$, implies that $N_{\overline{G}}(x) \cap S \neq N_{\overline{G}}(y) \cap S$. $\square$

Proposition 8. *For every pair (r, s), $r, s \in \mathbb{N}$, such that $3 \leq r$ and $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$, there exists a bipartite graph $G(r, s)$ such that $\lambda(\overline{G}) = \lambda(G) + 1$.*

Proof. Let $s = \left\lceil \frac{3r}{2} + 1 \right\rceil$. Consider the bipartite graph $G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ such that $V = U \cup W$, $U = [r] = \{1, 2, \ldots, r\}$, and $W \subseteq \mathcal{P}([r]) \setminus \{\emptyset\}$ is defined as follows. For $r = 2k$ even:

$$W = \Big\{[r]\Big\} \cup \Big\{[r] \setminus \{i\} : i \in [r]\Big\} \cup \Big\{[r] \setminus \{2i-1, 2i\} : 1 \leq i \leq k\Big\}$$

and for $r = 2k + 1$ odd:

$$\begin{aligned} W = &\Big\{[r]\Big\} \cup \Big\{[r] \setminus \{i\} : i \in [r]\Big\} \cup \Big\{[r] \setminus \{2i-1, 2i\} : 1 \leq i \leq k-1\Big\} \\ &\cup \Big\{[r] \setminus \{r-2, r-1\}, [r] \setminus \{r-1, , r\}, [r] \setminus \{r-2, r-1, r\}\Big\} \end{aligned}$$

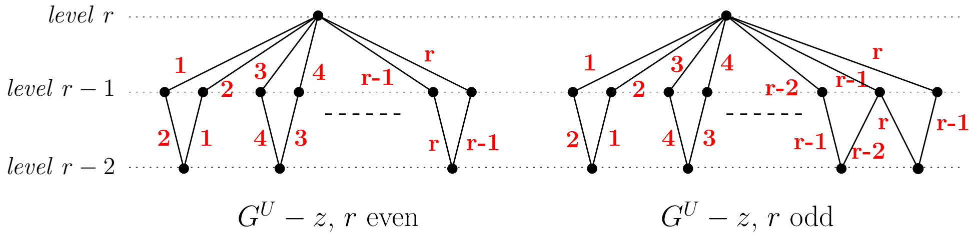


Figure 7: The labeled graph $G^U - z$, for $G = G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ and $U = \{1, \dots, r\}$.

By construction, U is an LD-set of G with r vertices and by Corollary 2, U is not an LD-set of $\overline{G}$ (see in Figure 7 the U-associated graph, G^U). We claim that there is no LD-set in $\overline{G}$ with at most r vertices.

Suppose that S is an LD-set of $\overline{G}$. We already know that $S \neq U$. Let us assume that $|S \cap U| = r - k$, $k \geq 1$. Consider the subgraph H of G^U induced by $2k$ edges of G^U with label $u \in U \setminus S$. Notice that, by definition, this subgraph exists and $z \notin V(H)$. Moreover, by Lemma 3, all connected components of H are cactus. Observe that, by definition of the associated graph G^U, the vertices lying at the same connected component of H have the same neighborhood in $S \cap U$. We know also that W induces a complete graph in $\overline{G}$. Therefore, at least all but one vertex of each connected component of H must be in S. By Lemma 4, this value is

$$|V(H)| - \mathrm{cc}(H) = \frac{3}{4}|E(H)| + \frac{1}{4}\mathrm{ex}(H) = \frac{3}{4}2k + \frac{1}{4}\mathrm{ex}(H) = \frac{3}{2}k + \frac{1}{4}\mathrm{ex}(H) \geq \frac{3}{2}k.$$

Hence, $|S| \geq (r - k) + \frac{3}{2}k = r + \frac{1}{2}k > r$.

Remark. We derive from this result that $\lambda(G) = r$. Nevertheless, a direct proof of this fact can be given: it can be proved in a similar way that there is no LD-set of G with less than r vertices.

For $s > \lceil \frac{3r}{2} + 1 \rceil$, we can add up to $2^r - 1 - r$ vertices to the set W of the graph $G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ taking into account that the neighborhoods in U of the vertices of W must be different and non-empty. □

Theorem 3. *Let r, s be a pair of integers such that $3 \leq r \leq s$.*

(1) There exists a bipartite graph $V(G) = U \cup W$ such that $|U| = r$, $|W| = s$ and $\lambda(\overline{G}) = \lambda(G) - 1$.

(2) There exists a bipartite graph $V(G) = U \cup W$ such that $|U| = r$, $|W| = s$ and $\lambda(\overline{G}) = \lambda(G)$.

(3) There exist a bipartite graph $V(G) = U \cup W$ such that $|U| = r$, $|W| = s$ and $\lambda(\overline{G}) = \lambda(G) + 1$ if and only if $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$.

Proof. To prove item (1), take the bi-star $K_2(r, s)$ and check that $\lambda(K_2(r, s)) = r + s - 2$ and $\lambda(\overline{K_2(r, s)}) = r + s - 3$. To prove item (2), take the biclique $K_{r,s}$ and check that $\lambda(K_{r,s}) = \lambda(\overline{K_{r,s}}) = r + s - 2$. Finally, observe that item (3) is a corollary of Propositions 6, 7 and Proposition 8. □

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